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General Relativity as a Conformally Invariant Scalar Gauge Field Theory

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The global symmetry implied by the fact that one can multiply all masses with a common constant is made into a local, gauge symmetry. The matter action then becomes conformally invariant and it seems natural to choose for the corresponding scalar gauge field the action for a conformally invariant (massless) scalar field. The resulting conformally invariant theory turns out to be equivalent to general relativity. Since this means that the usual Einstein–Hilbert action is not, in fact, a true gauge action for the space-time geometry, the full theory ought to be supplied with such a term. Gauge-theoretic arguments and conformal invariance requirements dictate its form.

But a truly infinitesimal geometry must recognize only the transference of length from one point to another point infinitely near the first. [H. Weyl (1918)]

1. INTRODUCTION

The title of this article may (should) seem absurd, since, as everyone concerned knows, general relativity (GR) is neither conformally invariant nor does it make use of a scalar field. Sometimes GR has been claimed to be a gauge theory, but that, too, is wrong (Yang, 1974). In this article I intend to show that GR in fact has all these qualities, though hidden in a subtle way. The present view of GR leads one in a natural way to thoughts on higher derivative corrections which constitute the conclusion of the article.

2. CONFORMAL SYMMETRY

The principle of equivalence states that all particles, under the influence of gravity, move along trajectories that are independent of the value of their

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mass. In prerelativistic mechanics one also noted that, as far as gravity is concerned, particle masses occur in such a way that nothing changes if they are all multiplied with a common constant; *only relative masses are of importance.* With modern gauge theories in mind, this can be viewed as a global symmetry of the Lagrangian, sometimes called scale invariance, and one is thus immediately led to considering making this symmetry local. This means that the multiplicative constant becomes a scalar field and the action for classical point particles with masses m_a becomes

$$S_m = \sum - \int m_a \phi \, ds \tag{1}$$

Here ϕ is the scalar field and ds is the element of length along the particle trajectory. The resulting action is now conformally invariant, i.e., invariant under transformations of the form

$$g_{ik}(x) \to g'_{ik}(x) = \Omega^2(x) g_{ik}(x) \tag{2}$$

where $g_{ik}(x)$ is the space-time metric and $\Omega^2(x)$ a so-called conformal factor. This assumes that the scalar field transforms in the standard way, i.e., according to

$$\phi(x) \to \phi'(x) = \Omega^{-1} \phi(x) \tag{3}$$

and the invariance then follows immediately from the definition of ds:

$$ds = (g_{ij} \, dx^i \, dx^j)^{1/2} \to ds' = \Omega \, ds \tag{4}$$

There is a multitude of reasons for assuming conformal invariance in physics, as originally pointed out by Weyl. Fundamentally, length is measured by placing an object adjacent to the measuring rod and comparing. Thus, the meaningfulness of comparing lengths of objects a finite distance apart must inevitably rest on some hypothesis about the nature of space-time. Another fact of relevance is that the causal null cone structure is invariant under conformal transformations (Penrose and Rindler, 1984).

3. THE SCALAR FIELD

In accordance with the standard prescription for the construction of gauge theories, we now need an action for the scalar field itself. It would seem silly to let this term ruin the invariance achieved in the matter term, so the only natural choice is the conformally invariant action given by

$$S_\phi = \int L_\phi(x) \, d^4x \tag{5}$$

where

$$L_\phi(x) = -\tfrac{1}{2}\sqrt{-g} \, (g^{ik} \phi_{,i} \phi_{,k} - \tfrac{1}{6} R \phi^2) \tag{6}$$

[see Wald (1984, Appendix 4), or Birrell and Davies (1982), whose sign conventions are used here]. The minus sign ensures the physically correct (positive) sign in front of the scalar curvature R. Note that the above construction is only natural in *four space-time dimensions*; the invariance of (6) is consistent with the transformation (3) only for this particular number of dimensions.

4. ON SCALAR-TENSOR THEORIES

One now notes that, for a given nonzero field ϕ, the conformal factor in (3) can always be chosen so that the transformed ϕ is constant. This means that, provided one can assume the field ϕ to be nonvanishing, the above theory is equivalent to GR. Harrison (1972) has considered a class of scalar-tensor theories, and the present theory is a member of this class. Of the theories in the Harrison class it is unique in being conformally invariant and in not giving any extra matter terms. More famous class members are the theories of Jordan and of Brans and Dicke. Harrison found that all these theories are, in fact, implicitly embodied in GR. The point of view arising from the present study turns this line of thought upside down: *General relativity is in fact a conformally invariant scalar field theory. The ordinary Einstein–Hilbert action arises for a particular choice of conformal factor corresponding to a particular choice of gauge.*

In the past, scalar-tensor theories were constructed by adding a scalar field to GR, but here this viewpoint is *reversed*: the scalar field is the starting point and GR results. Apart from the fact that GR thus has been made into a gauge theory, the present view is also pleasing since there are numerous motivations for a scalar field such as Mach's principle, Dirac's large number coincidence, etc., as discussed, for example, in Harrison (1972). However, one is now in a situation where the degrees of freedom corresponding to space-time geometry have no action of their own. The scalar curvature R *only appears indirectly* in the action (6) because of the requirement of conformal invariance.

5. HIGHER DERIVATIVE CORRECTIONS TO GR

Let us approach the question about the action for the geometry in the same spirit as above, i.e., through the analogy with gauge theories. These theories are characterized by the replacement of certain constants characterizing symmetries by fields and the simultaneous replacement of partial derivatives ∂_i with covariant derivatives D_i. The Lagrangian of the gauge field is then always something like $\sim[D_i, D_j]^2$, i.e., the square of the commutator of the covariant derivatives. In the case of gravitation these replacements, in the matter Lagrangian, are $\eta_{ik} \to g_{ik}(x)$ and $\partial_i \to \nabla_i = \partial_i + \Gamma_i$. The Lagrangian of the field itself should then be something like $\sim[\nabla_i, \nabla_j]^2 \sim R^2_{ijkl}$, as discussed by Yang (1974). Use of the Gauss–Bonnet theorem

$$\int (R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2)\sqrt{-g}\, d^4x = \text{Euler number} = \text{const} \tag{7}$$

however, makes such a Lagrangian equivalent to one expressed in terms of R^2_{ij} and R^2. This made Stelle (1978) investigate the two-parameter class of actions

$$S_\Gamma = \int \mathscr{L}_\Gamma(x)\, d^4x \tag{8}$$

with

$$\mathscr{L}_\Gamma(x) = \sqrt{-g}\,(\alpha R^{ij}R_{ij} - \beta R^2) \tag{9}$$

The R^2_{ijkl} action leads to $\alpha = 4\beta$, a combination of no particular significance otherwise. On the other hand, Stelle found that the combination $\alpha = 3\beta$ is special and corresponds to the higher derivative correction being purely repulsive in first approximation. Here I only wish to point out that this is the combination that arises from *conformal invariance.* That is, with Weyl's conformal curvature tensor one obtains

$$\mathscr{L}_\Gamma(x) = \sqrt{-g}\,\frac{\alpha}{2}\, C^{ijkl}C_{ijkl} \tag{10}$$

which is conformally invariant. Now the identity

$$C^{ijkl}C_{ijkl} = R^{ijkl}R_{ijkl} - 2R^{ij}R_{ij} + \tfrac{1}{3}R^2 \tag{11}$$

together with (7) shows that this is equivalent to the following Lagrangian:

$$\mathscr{L}_\Gamma(x) = \sqrt{-g}\,\alpha(R^{ij}R_{ij} - \tfrac{1}{3}R^2) \tag{12}$$

6. ON ORDERS OF MAGNITUDE

The term higher derivative correction used for the types of terms discussed above is, of course, not really appropriate; the gauge theory analogy indicates that the connection Γ corresponds to the potentials, and as indicated by the notation, the action (10) [or (12)] is in fact an ordinary action containing first derivatives of these potentials. Also in this respect ordinary GR is strange from the gauge theory point of view since there the potential is the metric and the connection is the force field. It is my hope that this article has succeeded in throwing some light on what lies behind this strangeness.

The full theory suggested here corresponds to a total action

$$S = S_m + S_\phi + S_\Gamma \tag{13}$$

and it is clear that in order for this to be physically realistic the constant value of the field ϕ must be proportional to the inverse of the gravitational constant G. Since it is reasonable to assume that the constant α in (12) is of order of magnitude unity, this yields a prediction of the relative importance of the higher derivative terms: if S_ϕ is of order unity (G^0), then S_m is of order G and S_Γ of order G^2. The new physical effects predicted by the theory outlined here are, of course, very small under ordinary conditions, but could be crucial in quantum gravity and cosmological circumstances.

7. CONCLUSIONS

It is instructive and a bit surprising to observe how the attempt to construct a conformally invariant scalar field theory forces one to include the scalar curvature in the action and how this term, so to speak, takes over the show. Apart from making GR into a gauge theory in a more natural way than before, the approach also leads one to higher derivative corrections and gives an estimate of their importance. Quite apart from the higher derivative discussions, the present approach provides a fairly convincing derivation of general relativity and thus shows why this theory is so effective and powerful.

REFERENCES

Birrell, N. D., and Davies, P. C. W. (1982). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge.

Harrison, E. R. (1972). *Physical Review D*, **6**, 2077.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, Vol. 1, Cambridge University Press, Cambridge.

Stelle, K. S. (1978). *General Relativity and Gravitation*, **9**, 353.

Wald, R. M. (1984). *General Relativity*, University of Chicago Press, Chicago, Illinois.

Yang, C. N. (1974). *Physical Review Letters*, **33**, 445.